

POISSON BOUNDARIES OVER LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. We present versions of several classical results on harmonic functions and Poisson boundaries in the setting of locally compact quantum groups \mathbb{G} . In particular, the Choquet–Deny theorem holds for compact quantum groups; also, the result of Kaimanovich–Vershik and Rosenblatt, which characterizes group amenability in terms of harmonic functions, answering a conjecture by Furstenberg, admits a non-commutative analogue in the separable case. We also explore the relation between classical and quantum Poisson boundaries by investigating the spectrum of the quantum group. We apply this machinery to find a concrete realization of the Poisson boundaries of the compact quantum group $SU_q(2)$ arising from measures on its spectrum. We further show that the Poisson boundary of the natural Markov operator extension of the convolution action of a quantum probability measure μ on $L_\infty(\mathbb{G})$ to $\mathcal{B}(L_2(\mathbb{G}))$, as introduced and studied – for general completely bounded multipliers on $L_1(\mathbb{G})$ – by M. Junge, M. Neufang and Z.-J. Ruan, can be identified precisely with the crossed product of the Poisson boundary of μ under the coaction of \mathbb{G} induced by the coproduct. This yields an affirmative answer, for general locally compact quantum groups, to a problem raised by M. Izumi (2004) in the commutative situation, in which he settled the discrete case, and unifies earlier results of W. Jaworski, M. Neufang and V. Runde.

1. INTRODUCTION

The theory of von Neumann algebras originated in a series of remarkable papers during the late 1930s and early 1940s by Murray and von Neumann. The theory may be viewed as an operator, or noncommutative, version of measure theory. During the last seventy years, operator algebras have proved to have a very profound structure theory. They also provide the foundation to consider the quantization of many areas of mathematics, such as analysis, topology, geometry, probability, and ergodic theory. Recently, work of Woronowicz, Baaĵ–Skandalis, and Kustermans–Vaes, has led to the very successful development of the theory of locally compact quantum groups. This provides the natural framework for the quantization of various problems related to groups and group actions on measure (or topological) spaces. The aim of this paper is to study Poisson boundaries over locally compact quantum groups.

Poisson boundaries and harmonic functions have played a very important role in the study of random walks on discrete groups, and more generally in harmonic analysis and ergodic theory on locally compact groups (see for instance Furstenberg’s seminal work [13]). Let us recall that if G is a locally compact group and μ is a probability measure on G , we obtain a Markov operator Φ_μ on $L_\infty(G)$ associated with

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the measure μ which is defined by

$$(1.1) \quad \Phi_\mu(h)(s) = \mu \star h(s) = \int_G h(st) d\mu(t) \quad (s \in G).$$

It is known that there exists a probability measure space (Π, ν) , the Poisson boundary of (G, μ) , such that $L_\infty(\Pi, \nu)$ can be identified with the weak* closed subspace \mathcal{H}_μ of $L_\infty(G)$ which consists of all μ -harmonic functions, i.e., functions h on G satisfying $\Phi_\mu(h) = h$.

The noncommutative version of this concept can be considered in two different directions: one replaces $L_\infty(G)$ by certain quantum groups, such as its dual (quantum) group von Neumann algebra $VN(G)$ or quantum groups arising from mathematical physics; in another direction, one can perform the quantum mechanical passage from $L_\infty(G)$ to $\mathcal{B}(L_2(G))$.

Poisson boundaries over discrete quantum groups were first studied by Izumi [19]. He used these objects to study compact quantum group actions. More precisely, he showed that the relative commutant of the fixed point algebra of certain ITP actions of compact quantum groups can be realized as the Poisson boundary over the dual (discrete) quantum group. As a concrete example, he showed that in the case of Woronowicz' compact quantum group $SU_q(2)$, this gives an identification between the Poisson boundary of a Markov operator on the dual quantum group, with one of the Podleś spheres [33]. Later this result was generalized to the case of $SU_q(n)$ by Izumi, Neshveyev and Tuset [21]. Poisson boundaries for other discrete quantum groups have been studied by Vaes, Vander Vennet and Vergnioux [37], [38], [39].

On the other hand, Jaworski and Neufang [22] investigated the second way of quantization. Their starting point is a result by Ghahramani [14] stating that there is a natural isometric representation Θ of the measure algebra $M(G)$ as operators acting on $\mathcal{B}(L_2(G))$, such that for $\mu \in M(G)$ and $h \in L_\infty(G)$ – viewed as a multiplication operator on $L_2(G)$ – we have $\Theta(\mu)(h) = \mu \star h$. So, they replaced the commutative von Neumann algebra $L_\infty(G)$ by $\mathcal{B}(L_2(G))$, where the latter is endowed with the extended convolution action given by Θ . The dual analogue of this representation has been studied by Neufang, Ruan and Spronk [31] for the action of the Fourier–Stieltjes algebra $B(G)$ on $VN(G)$, extended to $\mathcal{B}(L_2(G))$. Subsequently, Neufang and Runde [32] used this (dual) extension theorem to combine the two types of quantizations in this case, and studied the harmonic operators in $\mathcal{B}(L_2(G))$ with respect to the extended action of a positive definite function.

In this paper, we establish important classical results on Poisson boundaries and harmonic functions in the quantum group setting: first concerning harmonic operators on the level of the quantum group $L_\infty(\mathbb{G})$, and finally the quantization to the level of $\mathcal{B}(L_2(\mathbb{G}))$. Indeed, in the final section we prove a result for the space of harmonic operators in $\mathcal{B}(L_2(\mathbb{G}))$ in terms of the Poisson boundary at the level of $L_\infty(\mathbb{G})$. The general framework for our study are thus Markov operators (i.e., normal, completely positive, unital maps) stemming from quantum group actions.

The paper is organized as follows. We recall relevant definitions and introduce some notation in section 2. In section 3, we establish quantized versions of several classical results concerning Poisson boundaries. In particular, we prove that the Poisson boundary of a non-degenerate quantum ‘probability measure’ on

a locally compact quantum group is never a subalgebra, unless trivial, and that there is no non-trivial harmonic operator which is ‘*continuous*’ and ‘*vanishing at infinity*’.

In the classical setting there is a characterization of amenability of a (σ -compact) locally compact group in terms of its Poisson boundaries. Kaimanovich–Vershik [24] and Rosenblatt [34] independently proved that if G is a (σ -compact) locally compact amenable group, then there exists an absolutely continuous measure μ on G (i.e., $\mu \in L_1(G)$) such that μ -harmonic functions are trivial. This answered a conjecture of Furstenberg [13], who had shown the converse. In section 4 we prove the corresponding result in the quantum setting (Theorem 4.2).

The classical Choquet–Deny theorem [4] states that there is no non-trivial μ -harmonic function for an adapted probability measure on a locally compact abelian group G . The conclusion of this theorem has been proved for many other cases, including compact groups. In section 5 we study Poisson boundaries over compact quantum groups, and we prove a noncommutative version of the Choquet–Deny theorem in this setting (Theorem 5.3).

We investigate the relation between the classical and the quantum setting in section 6, by proving a formula which links the Poisson boundary of a Markov operator induced from a commutative quantum subgroup, to its classical counterpart. Applying our machinery to the case of $SU_q(2)$, we show that the Poisson boundary over the latter, induced from the quantum subgroup \mathbb{T} , i.e., the spectrum of $SU_q(2)$, can be identified with a Podleś sphere.

In section 7, we consider the natural extension of the action of a quantum probability measure μ on $L_\infty(\mathbb{G})$ to $\mathcal{B}(L_2(\mathbb{G}))$, as introduced and studied in [23]. We completely describe the structure of the Poisson boundary $\mathcal{H}_{\Theta(\mu)}$ of the induced Markov operator $\Theta(\mu)$. The main result (Theorem 7.5) in this section gives a concrete realization of the Poisson boundary $\mathcal{H}_{\Theta(\mu)}$ as the crossed product of \mathcal{H}_μ with \mathbb{G} under a natural action. This generalizes and unifies the corresponding results concerning commutative, respectively, cocommutative quantum groups, as obtained in [22, Proposition 6.3] and [32, Theorem 4.8 (ii)]. Note that [22, Proposition 6.3] gave an affirmative answer to a problem raised by Izumi [20, Problem 4.3]. Hence, the present paper answers Izumi’s question in the situation of locally compact quantum groups. The main result (Theorem 4.1) of [20], which gives a crossed product formula for Poisson boundaries associated with $L_\infty(G)$, where G is a countable discrete group, follows from our theorem applied to discrete commutative quantum groups. Our result also implies that the quantum group crossed product $\mathbb{G} \rtimes_{\Gamma_\mu} \mathcal{H}_\mu$ is always an injective von Neumann algebra. Applied to the case $L_\infty(G)$ for a locally compact group G , this immediately yields Zimmer’s classical result that the natural action of G on its Poisson boundaries is always amenable.

2. DEFINITIONS AND PRELIMINARY RESULTS

In this paper, we denote by $\mathbb{G} = (L_\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ a (von Neumann algebraic) *locally compact quantum group* in the sense of Kustermans and Vaes [27], [28]. The *right Haar weight* ψ (which is an n.s.f.

right invariant weight) on the quantum group von Neumann algebra $L_\infty(\mathbb{G})$ determines a Hilbert space $L_2(\mathbb{G}) = L_2(\mathbb{G}, \psi)$ and we obtain the *right fundamental unitary operator* V on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$, which satisfies the *pentagonal relation*

$$(2.1) \quad V_{12}V_{13}V_{23} = V_{23}V_{12}.$$

Here we used the leg notation $V_{12} = V \otimes 1$, $V_{23} = 1 \otimes V$, and $V_{13} = (\iota \otimes \chi)V_{12}$, where $\chi(x \otimes y) = y \otimes x$ is the flip map. This fundamental unitary operator induces a coassociative comultiplication

$$(2.2) \quad \tilde{\Gamma} : x \in \mathcal{B}(L_2(\mathbb{G})) \rightarrow \tilde{\Gamma}(x) = V(x \otimes 1)V^* \in \mathcal{B}(L_2(\mathbb{G}) \otimes L_2(\mathbb{G}))$$

on $\mathcal{B}(L_2(\mathbb{G}))$, for which we have $\tilde{\Gamma}|_{L_\infty(\mathbb{G})} = \Gamma$.

Let $L_1(\mathbb{G})$ be the predual of $L_\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces an associative completely contractive multiplication

$$(2.3) \quad \star : f \otimes g \in L_1(\mathbb{G}) \hat{\otimes} L_1(\mathbb{G}) \rightarrow f \star g = (f \otimes g)\Gamma \in L_1(\mathbb{G})$$

on $L_1(\mathbb{G})$. Since the multiplication \star is a complete quotient map from $L_1(\mathbb{G}) \hat{\otimes} L_1(\mathbb{G})$ onto $L_1(\mathbb{G})$, we get

$$(2.4) \quad L_1(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = \overline{\text{span}\{f \star g : f, g \in L_1(\mathbb{G})\}}^{\|\cdot\|}.$$

If $\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a, \psi_a)$ is the commutative quantum group associated with a locally compact group G , then $L_1(\mathbb{G}_a)$ is just the convolution algebra $L_1(G)$. If on the other hand $\mathbb{G}_s = \hat{\mathbb{G}}_a$ is the cocommutative dual quantum group of \mathbb{G}_a , then $L_1(\mathbb{G}_s)$ is the Fourier algebra $A(G)$.

The *right regular representation* $\rho : L_1(\mathbb{G}) \rightarrow \mathcal{B}(L_2(\mathbb{G}))$ is defined by

$$\rho : f \in L_1(\mathbb{G}) \rightarrow \rho(f) = (\iota \otimes f)(V) \in \mathcal{B}(L_2(\mathbb{G})),$$

which is an injective and completely contractive algebra homomorphism from $L_1(\mathbb{G})$ into $\mathcal{B}(L_2(\mathbb{G}))$. We let $L_\infty(\hat{\mathbb{G}}') = \{\rho(f) : f \in L_1(\mathbb{G})\}''$ denote the quantum group von Neumann algebra of the (commutant) dual quantum group $\hat{\mathbb{G}}'$. Then $\hat{V} = \Sigma V^* \Sigma$, where Σ denotes the *flip operator* $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$, is the right fundamental unitary operator of $\hat{\mathbb{G}}'$, and

$$(2.5) \quad \hat{\rho} : \hat{f}' \in L_1(\hat{\mathbb{G}}') \rightarrow \hat{\rho}(\hat{f}') = (\iota \otimes \hat{f}')(\hat{V}) = (\hat{f}' \otimes \iota)(V^*) \in L_\infty(\mathbb{G})$$

is the right regular representation of $\hat{\mathbb{G}}'$. The *reduced quantum group C^* -algebra* $C_0(\mathbb{G}) = \overline{\hat{\rho}(L_1(\hat{\mathbb{G}}'))}^{\|\cdot\|}$ is a weak* dense C^* -subalgebra of $L_\infty(\mathbb{G})$ with the comultiplication

$$\Gamma : C_0(\mathbb{G}) \rightarrow M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$$

given by the restriction of the comultiplication of $L_\infty(\mathbb{G})$ to $C_0(\mathbb{G})$. Here, $M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ denotes the multiplier C^* -algebra of the minimal C^* -algebra tensor product $C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$. For convenience, we often use $C(\mathbb{G})$ for $M(C_0(\mathbb{G}))$. Let $M(\mathbb{G})$ denote the operator dual $C_0(\mathbb{G})^*$. There exists a completely contractive multiplication on $M(\mathbb{G})$ given by the convolution

$$\star : \mu \otimes \nu \in M(\mathbb{G}) \hat{\otimes} M(\mathbb{G}) \mapsto \mu \star \nu = \mu(\iota \otimes \nu)\Gamma = \nu(\mu \otimes \iota)\Gamma \in M(\mathbb{G})$$

such that $M(\mathbb{G})$ contains $L_1(\mathbb{G})$ as a norm closed two-sided ideal (for details see [3], [16] and [18]).

Let $L_{1*}(\hat{\mathbb{G}}') = \{\hat{\omega}' \in L_1(\hat{\mathbb{G}}') : \exists \hat{f}' \in L_1(\hat{\mathbb{G}}') \text{ such that } \hat{\rho}(\hat{\omega}')^* = \hat{\rho}(\hat{f}')\}$. Then $L_{1*}(\hat{\mathbb{G}}') \subseteq L_1(\hat{\mathbb{G}}')$ is norm dense, and with the involution $(\hat{\omega}')^* = \hat{f}'$, and the norm $\|\hat{\omega}'\|_u = \max\{\|\hat{\omega}'\|, \|(\hat{\omega}')^*\|\}$, the space $L_{1*}(\hat{\mathbb{G}}')$ becomes a Banach $*$ -algebra (see [26] for details). We obtain the *universal quantum group C^* -algebra* $C_u(\mathbb{G})$ as the universal enveloping C^* -algebra of the Banach algebra $L_{1*}(\hat{\mathbb{G}}')$. There is a universal $*$ -representation

$$\hat{\rho}_u : L_{1*}(\hat{\mathbb{G}}') \rightarrow \mathcal{B}(H_u)$$

such that $C_u(\mathbb{G}) = \overline{\hat{\rho}_u(L_{1*}(\hat{\mathbb{G}}'))}^{\|\cdot\|}$. There is a universal comultiplication

$$\Gamma_u : C_u(\mathbb{G}) \rightarrow M(C_u(\mathbb{G}) \otimes C_u(\mathbb{G})),$$

and the operator dual $M_u(\mathbb{G}) = C_u(\mathbb{G})^*$, which can be regarded as the space of all *quantum measures* on \mathbb{G} , is a unital completely contractive Banach algebra with multiplication given by

$$\omega \star_u \mu = \omega(\iota \otimes \mu) \Gamma_u = \mu(\omega \otimes \iota) \Gamma_u$$

(see [2], [3] and [26]). By the universal property of $C_u(\mathbb{G})$, there is a unique surjective $*$ -homomorphism $\pi : C_u(\mathbb{G}) \rightarrow C_0(\mathbb{G})$ such that $\pi(\hat{\rho}_u(\hat{\omega}')) = \hat{\rho}(\hat{\omega}')$ for all $\hat{\omega}' \in L_{1*}(\hat{\mathbb{G}}')$. Moreover, the adjoint map $\pi^* : M(\mathbb{G}) \rightarrow M_u(\mathbb{G})$ defines a completely isometric injection such that $\omega \star_u \pi^*(\mu)$ and $\pi^*(\mu) \star_u \omega$ are in $\pi^*(M(\mathbb{G}))$ for all $\omega \in M_u(\mathbb{G})$ and $\mu \in M(\mathbb{G})$ (cf. [26, Proposition 6.2]). Therefore we can identify $M(\mathbb{G})$ with a norm closed two-sided ideal in $M_u(\mathbb{G})$, and

$$(2.6) \quad \omega \star \mu = (\pi^*)^{-1}(\omega \star_u \pi^*(\mu)) \in M(\mathbb{G}) \quad , \quad \mu \star \omega = (\pi^*)^{-1}(\pi^*(\mu) \star_u \omega) \in M(\mathbb{G})$$

define actions of $M_u(\mathbb{G})$ on $M(\mathbb{G})$. In particular, the restriction of π^* to $L_1(\mathbb{G})$ is a completely isometric injection from $L_1(\mathbb{G})$ into $M_u(\mathbb{G})$. Since $L_1(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle$, we can conclude that $\omega \star f$ and $f \star \omega$ are again contained in $L_1(\mathbb{G})$. Therefore, we can also identify $L_1(\mathbb{G})$ with a norm closed two-sided ideal in $M_u(\mathbb{G})$.

A linear map m on $L_1(\mathbb{G})$ is called a *right (respectively, left) centralizer* of $L_1(\mathbb{G})$ if it satisfies

$$m(f \star g) = f \star m(g) \text{ (respectively, } m(f \star g) = m(f) \star g)$$

for all $f, g \in L_1(\mathbb{G})$. A pair of maps (m_1, m_2) on $L_1(\mathbb{G})$ is a *double centralizer* if $f \star m_1(g) = m_2(f) \star g$. If (m_1, m_2) is a double centralizer of $L_1(\mathbb{G})$, then m_1 is a left centralizer and m_2 is a right centralizer of $L_1(\mathbb{G})$. In this paper, we are mainly interested in right centralizers and we denote by $C_{cb}^r(L_1(\mathbb{G}))$ the space of all completely bounded right centralizers of $L_1(\mathbb{G})$. The results for left centralizers and double centralizers can be obtained analogously.

It is seen from (2.6) that for each $\omega \in M_u(\mathbb{G})$, we obtain a pair of completely bounded maps

$$(2.7) \quad \mathbf{m}_\omega^l(f) = \omega \star f \text{ and } \mathbf{m}_\omega^r(f) = f \star \omega$$

on $L_1(\mathbb{G})$ with $\max\{\|\mathbf{m}_\omega^l\|_{cb}, \|\mathbf{m}_\omega^r\|_{cb}\} \leq \|\omega\|$. It turns out that this pair of maps $(\mathbf{m}_\omega^l, \mathbf{m}_\omega^r)$ is a double centralizer of $L_1(\mathbb{G})$. Therefore, we obtain the natural inclusions

$$(2.8) \quad L_1(\mathbb{G}) \hookrightarrow M(\mathbb{G}) \hookrightarrow M_u(\mathbb{G}) \hookrightarrow C_{cb}^r(L_1(\mathbb{G})),$$

where the first two inclusions are completely isometric homomorphisms, and the last one is a completely contractive homomorphism. These algebras are typically not equal. We have

$$(2.9) \quad M(\mathbb{G}) = M_u(\mathbb{G}) = C_{cb}^r(L_1(\mathbb{G}))$$

if and only if \mathbb{G} is co-amenable, i.e., $L_1(\mathbb{G})$ has a contractive (or bounded) approximate identity (cf. [2], [3], and [18]). If \mathbb{G}_a is the commutative quantum group associated with a locally compact group G , then \mathbb{G}_a is always co-amenable since $L_1(\mathbb{G}_a) = L_1(G)$ has a contractive approximate identity. On the other hand, if $\mathbb{G}_s = \hat{\mathbb{G}}_a$ is the cocommutative dual quantum group of \mathbb{G}_a , it is co-amenable, i.e., the Fourier algebra $A(G)$ has a contractive approximate identity, if and only if the group G is amenable.

Now, given any $\mu \in M_u(\mathbb{G})$ (respectively, $\mu \in M(\mathbb{G})$), we obtain the completely bounded right centralizer $\mathbf{m}_\mu^r \in C_{cb}^r(L_1(\mathbb{G}))$. The adjoint map $\Phi_\mu = (\mathbf{m}_\mu^r)^*$ is a normal completely bounded map on $L_\infty(\mathbb{G})$ such that $\Phi_\mu(x) = \mu \star x$, more precisely,

$$(2.10) \quad \langle f, \Phi_\mu(x) \rangle = \langle f \star \mu, x \rangle = \langle f, \mu \star x \rangle$$

for all $x \in L_\infty(\mathbb{G})$ and $f \in L_1(\mathbb{G})$. Moreover, the map Φ_μ satisfies the covariance condition

$$(2.11) \quad \Gamma \circ \Phi_\mu = (\iota \otimes \Phi_\mu) \circ \Gamma,$$

or equivalently, $\Phi_\mu(x \star f) = \Phi_\mu(x) \star f$ for all $x \in L_\infty(\mathbb{G})$ and $f \in L_1(\mathbb{G})$. We are particularly interested in the case when μ is a state in $M_u(\mathbb{G})$. In this case, Φ_μ is unital completely positive, i.e. a Markov operator, on $L_\infty(\mathbb{G})$. We note, even though we do not need this fact, that if $m \in C_{cb}^r(L_1(\mathbb{G}))$ is a completely bounded right centralizer such that Φ_m is a Markov operator on $L_\infty(\mathbb{G})$, then we must have $m = m_\mu$ for some state $\mu \in M_u(\mathbb{G})$ (see[6]). If we let

$$LUC(\mathbb{G}) = \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = \overline{\text{span}\{x \star f : x \in L_\infty(\mathbb{G}), f \in L_1(\mathbb{G})\}}^{\|\cdot\|}$$

denote the space of *left uniformly continuous linear functionals* on $L_1(\mathbb{G})$, then Φ_μ maps $LUC(\mathbb{G})$ into $LUC(\mathbb{G})$. Under certain conditions, $LUC(\mathbb{G})$ is a unital C^* -subalgebra of $C(\mathbb{G})$ (cf. [17], [18] and [36]). Also, since Φ_μ maps $C_0(\mathbb{G})$ into $C_0(\mathbb{G})$, if Φ_μ is completely positive, it maps $C(\mathbb{G})$ into $C(\mathbb{G})$ (see [29]). In general, we have

$$C_0(\mathbb{G}) \subseteq LUC(\mathbb{G}) \subseteq C(\mathbb{G}) \subseteq L_\infty(\mathbb{G}).$$

In particular, if \mathbb{G} is a compact quantum group, we have $C_0(\mathbb{G}) = LUC(\mathbb{G}) = C(\mathbb{G})$, and if \mathbb{G} is discrete, we have $LUC(\mathbb{G}) = C(\mathbb{G}) = L_\infty(\mathbb{G})$.

3. POISSON BOUNDARIES OF QUANTUM MEASURES

We denote by $\mathcal{P}_u(\mathbb{G})$ the set of all states on $C_u(\mathbb{G})$ (i.e., ‘the quantum probability measures’). Then Φ_μ is a *Markov operator*, i.e., a unital normal completely positive map, on $L_\infty(\mathbb{G})$.

We consider the space of fixed points $\mathcal{H}^\mu = \{x \in L_\infty(\mathbb{G}) : \Phi_\mu(x) = x\}$. It is easy to see that \mathcal{H}^μ is a weak* closed operator system in $L_\infty(\mathbb{G})$. In fact, we obtain a natural von Neumann algebra product on this space. Let us recall this construction for the convenience of the reader (cf. [19, Section 2.5]).

We first define a projection $\mathcal{E}_\mu : L_\infty(\mathbb{G}) \rightarrow L_\infty(\mathbb{G})$ of norm one by taking the weak* limit

$$(3.1) \quad \mathcal{E}_\mu(x) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \Phi_\mu^k(x)$$

with respect to a free ultrafilter \mathcal{U} on \mathbb{N} . It is easy to see that $\mathcal{H}^\mu = \mathcal{E}_\mu(L_\infty(\mathbb{G}))$, and that the Choi-Effros product

$$(3.2) \quad x \circ y = \mathcal{E}_\mu(xy)$$

defines a von Neumann algebra product on \mathcal{H}^μ . We note that this product is independent of the choice of the free ultrafilter \mathcal{U} since every completely positive isometric linear isomorphism between two von Neumann algebras is a *-isomorphism. To avoid confusion, we denote by $\mathcal{H}_\mu = (\mathcal{H}^\mu, \circ)$ this von Neumann algebra, and we call \mathcal{H}_μ the *Poisson boundary* of μ .

Our goal in this section is to prove quantum versions of several important results which are well-known in the classical setting. In order to prove our results in a general form for the Markov operators corresponding to states on the universal C^* -algebra $C_u(\mathbb{G})$, rather than just the ones in $M(\mathbb{G})$, we need to work with the universal von Neumann algebra $C_u(\mathbb{G})^{**}$. But there are some technical difficulties that arise in the non-Kac setting if one wants to lift all quantum group properties to the universal von Neumann algebra (cf. [26]). So in the following we make sure that the properties we need for our purpose are all valid at the universal von Neumann algebraic level.

Since $L_1(\mathbb{G})$ is a norm closed two-sided ideal in $M_u(\mathbb{G})$, we obtain a natural $M_u(\mathbb{G})$ -bimodule structure on $L_1(\mathbb{G})$, and its adjoint defines an $M_u(\mathbb{G})$ -bimodule structure on $L_\infty(\mathbb{G})$ such that

$$\langle f, \mu \star x \rangle = \langle f \star \mu, x \rangle \quad \text{and} \quad \langle f, x \star \mu \rangle = \langle \mu \star f, x \rangle$$

for all $f \in L_1(\mathbb{G})$, $\mu \in M_u(\mathbb{G})$ and $x \in L_\infty(\mathbb{G})$. On the other hand, there is a natural $M_u(\mathbb{G})$ -bimodule structure on the universal enveloping von Neumann algebra $C_u(\mathbb{G})^{**}$ given by

$$\langle \omega, \mu \star_u x_u \rangle = \langle \omega \star_u \mu, x_u \rangle \quad \text{and} \quad \langle \omega, x_u \star_u \mu \rangle = \langle \mu \star_u \omega, x_u \rangle$$

for all $\omega, \mu \in M_u(\mathbb{G})$ and $x_u \in C_u(\mathbb{G})^{**}$. Let us denote by $\tilde{\pi} = (\pi^*|_{L_1(\mathbb{G})})^* : C_u(\mathbb{G})^{**} \rightarrow L_\infty(\mathbb{G})$ the normal surjective *-homomorphism extension of π to $C_u(\mathbb{G})^{**}$. We obtain the following interesting connection

$$(3.3) \quad \tilde{\pi}(\mu \star_u x_u) = \mu \star \tilde{\pi}(x_u) \quad \text{and} \quad \tilde{\pi}(x_u \star_u \mu) = \tilde{\pi}(x_u) \star \mu$$

between the two module structures. Indeed, for any $x_u \in C_u(\mathbb{G})^{**}$, we deduce from (2.6) that

$$\langle f, \mu \star \tilde{\pi}(x_u) \rangle = \langle f \star \mu, \tilde{\pi}(x_u) \rangle = \langle \pi^*(f) \star_u \mu, x_u \rangle = \langle \pi^*(f), \mu \star_u x_u \rangle = \langle f, \tilde{\pi}(\mu \star_u x_u) \rangle$$

for all $f \in L_1(\mathbb{G})$ and $\mu \in M_u(\mathbb{G})$.

The following result extends [26, Proposition 6.2] to the von Neumann algebraic level. Here we denote by \mathcal{V} the *universal left regular corepresentation* of $C_0(\mathbb{G})$ considered in [26, Proposition 5.1].

Proposition 3.1. *For any $x_u \in C_u(\mathbb{G})^{**}$ and $f \in L_1(\mathbb{G})$ (or $f \in M(\mathbb{G})$), we have*

$$\pi^*(f) \star_u x_u = (\iota \otimes f) \mathcal{V}^*(1 \otimes \tilde{\pi}(x_u)) \mathcal{V}.$$

Proof. Given $x_u \in C_u(\mathbb{G})^{**}$, there exists a net of elements $x_i \in C_u(\mathbb{G})$ such that $\|x_i\| \leq \|x_u\|$ and $x_i \rightarrow x_u$ in the weak* topology. It is known from [26, Proposition 6.2] that for each $x_i \in C_u(\mathbb{G})$, we have

$$(3.4) \quad (\iota \otimes \pi) \Gamma_u(x_i) = \mathcal{V}^*(1 \otimes \pi(x_i)) \mathcal{V}.$$

Therefore, we get

$$\begin{aligned} \langle \mu, \pi^*(f) \star_u x_u \rangle &= \langle \mu \star_u \pi^*(f), x_u \rangle = \lim \langle \mu \star_u \pi^*(f), x_i \rangle \\ &= \lim \langle \mu \otimes \pi^*(f), \Gamma_u(x_i) \rangle = \lim \langle \mu \otimes f, \mathcal{V}^*(1 \otimes \pi(x_i)) \mathcal{V} \rangle \\ &= \langle \mu, (\iota \otimes f) \mathcal{V}^*(1 \otimes \tilde{\pi}(x_u)) \mathcal{V} \rangle \end{aligned}$$

for all $\mu \in M_u(\mathbb{G})$ and $f \in L_1(\mathbb{G})$ (or $f \in M(\mathbb{G})$). \square

Using Proposition 3.1, we can prove the following result; the idea of the proof is similar to the proof of [36, Theorem 2.4].

Proposition 3.2. *For any $f \in L_1(\mathbb{G})$ and $x_u \in C_u(\mathbb{G})^{**}$, both $\pi^*(f) \star_u x_u$ and $x_u \star_u \pi^*(f)$ are in $M(C_u(\mathbb{G}))$.*

Proof. For $f \in L_1(\mathbb{G})$, we can write $f = y' \cdot f'$ for some $y' \in \mathcal{K}(L_2(\mathbb{G}))$ and $f' \in L_1(\mathbb{G})$, where $\mathcal{K}(L_2(\mathbb{G}))$ denotes the C^* -algebra of all compact operators on the Hilbert space $L_2(\mathbb{G})$, and \cdot is the canonical action of $\mathcal{K}(L_2(\mathbb{G}))$ on its dual. Since $\mathcal{V} \in M(C_u(\mathbb{G}) \otimes \mathcal{K}(L_2(\mathbb{G})))$ (see [26, Proposition 5.1]), we have

$$(\pi^*(f) \star_u x_u) a = \langle \iota \otimes f', \mathcal{V}^*(1 \otimes \tilde{\pi}(x_u)) \mathcal{V}(a \otimes y') \rangle \in C_u(\mathbb{G})$$

for all $a \in C_u(\mathbb{G})$. Here we used the fact that $\mathcal{V}^*(1 \otimes \tilde{\pi}(x_u)) \mathcal{V}(a \otimes y') \in C_u(\mathbb{G}) \otimes \mathcal{K}(L_2(\mathbb{G}))$. This shows that $\pi^*(f) \star_u x_u \in M(C_u(\mathbb{G}))$. Similarly, we can prove that $x_u \star_u \pi^*(f) \in M(C_u(\mathbb{G}))$ by considering the *universal right regular corepresentation* of $C_0(\mathbb{G})$. \square

In the classical setting, when considering the Poisson boundaries and harmonic functions on a locally compact group G , in order to rule out trivialities, one usually works with measures whose support generates G as a closed semigroup or group. Therefore it is natural to seek for a quantum version of such a property and restrict ourselves to those *quantum measures* possessing that property.

A state $\mu \in \mathcal{P}_u(\mathbb{G})$ is called *non-degenerate* on $C_u(\mathbb{G})$ if for every non-zero element $x_u \in C_u(\mathbb{G})^+$, there exists $n \in \mathbb{N}$ such that $\langle x_u, \mu^n \rangle \neq 0$ (see also [39, Terminology 5.4]). Non-degeneracy can be defined similarly for states $\mu \in M(\mathbb{G})$ on $C_0(\mathbb{G})$. Note that every faithful state is non-degenerate, but there are examples of non-faithful non-degenerate states. If $\mu \in \mathcal{P}_u(\mathbb{G})$, then there exists a unique strictly continuous state extension of μ to a state on $M(C_u(\mathbb{G}))$, which we still denote by μ .

Lemma 3.3. *Let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate. Then for every non-zero $x_u \in M(C_u(\mathbb{G}))^+$, there exists $n \in \mathbb{N}$ such that $\langle x_u, \mu^n \rangle \neq 0$.*

Proof. Let $x_u \in M(C_u(\mathbb{G}))^+$ be non-zero, and let $a_u \in C_u(\mathbb{G})^+$ be such that $\|a_u\| = 1$ and $a_u^{\frac{1}{2}} x_u^{\frac{1}{2}} \neq 0$. Then we have $C_u(\mathbb{G})^+ \ni x_u^{\frac{1}{2}} a_u x_u^{\frac{1}{2}} \leq x_u$. Now since μ is non-degenerate, there exists $n \in \mathbb{N}$ such that $0 < \langle x_u^{\frac{1}{2}} a_u x_u^{\frac{1}{2}}, \mu^n \rangle \leq \langle x_u, \mu^n \rangle$. \square

Lemma 3.4. *Let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate, and let ψ be the right Haar weight of \mathbb{G} . Then Φ_μ is ψ -invariant and thus faithful on $L_\infty(\mathbb{G})$.*

Proof. Since ψ is the right Haar weight of \mathbb{G} , we have

$$\psi(\Phi_\mu(x))1 = (\psi \otimes \iota)\Gamma(\Phi_\mu(x)) = (\psi \otimes \Phi_\mu)\Gamma(x) = \Phi_\mu((\psi \otimes \iota)\Gamma(x)) = \psi(x)1$$

for all $x \in L_\infty(\mathbb{G})^+$. This implies $\psi \circ \Phi_\mu = \psi$ on $L_\infty(\mathbb{G})^+$ and thus Φ_μ is faithful on $L_\infty(\mathbb{G})$. \square

The following lemma is essential for our results concerning non-degenerate states.

Lemma 3.5. *Let \mathbb{G} be a locally compact quantum group and let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate. Let $x \in L_\infty(\mathbb{G})$ be a self-adjoint element which attains its norm on $L_1(\mathbb{G})_1^+$. If $x \in \mathcal{H}_\mu$ then $x \in \mathbb{C}1$.*

Proof. Suppose that $\|x\| = 1$ and $f \in L_1(\mathbb{G})^+$ is a state such that $\langle f, x \rangle = 1$. Now assume towards a contradiction that $x \neq 1$. Then $1 - x$ is a non-zero positive element in $L_\infty(\mathbb{G}) \cap \mathcal{H}_\mu$ and so there exists a non-zero positive element $x_u \in C_u(\mathbb{G})^{**}$ such that $\tilde{\pi}(x_u) = 1 - x$. Then by Proposition 3.2, we have $x_u \star_u \pi^*(f) \in M(C_u(\mathbb{G}))$. Moreover, by (3.3) we have $\tilde{\pi}(x_u \star_u \pi^*(f)) = (1 - x) \star f$. It follows from Lemma 3.4 that $(1 - x) \star f$ is non-zero, which implies that $x_u \star_u \pi^*(f) \in M(C_u(\mathbb{G}))$ is a non-zero positive element. Since μ is non-degenerate, by Lemma 3.3, there exists $n \in \mathbb{N}$ such that

$$\langle 1 - x, f \star \mu^n \rangle = \langle x_u \star_u \pi^*(f), \mu^n \rangle \neq 0.$$

On the other hand, since $x \in \mathcal{H}_\mu$ we have $\Phi_{\mu^n}(x) = x$. It follows that

$$\langle 1, f \star \mu^n \rangle = 1 = \langle x, f \rangle = \langle \Phi_{\mu^n}(x), f \rangle = \langle x, f \star \mu^n \rangle.$$

This implies that $\langle 1 - x, f \star \mu^n \rangle = 0$, which is a contradiction. Hence, we must have $x = 1$. \square

If μ is a non-degenerate probability measure on a locally compact group G , it is well-known that the space of all μ -harmonic functions is never a subalgebra of $L_\infty(G)$, unless trivial. Using the previous lemma, we can prove a quantum version of this result.

Theorem 3.6. *Let \mathbb{G} be a locally compact quantum group and let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate. Then the following are equivalent:*

- (i) \mathcal{H}_μ is a subalgebra of $L_\infty(\mathbb{G})$;
- (ii) $\mathcal{H}_\mu = \mathbb{C}1$.

Proof. We just need to prove (i) \Rightarrow (ii). Since \mathcal{H}_μ is a weak* closed operator system, (i) implies that \mathcal{H}_μ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$, and is therefore generated by its projections. Now let $0 \neq p \in \mathcal{H}_\mu$ be a projection and $\xi \in L_2(\mathbb{G})$ a unit vector such that $p\xi = \xi$. Then we have $\|p\| = 1 = \langle p\xi, \xi \rangle$, which shows that p attains its norm on $L_1(\mathbb{G})_1^+$. Hence, $p = 1$ by Lemma 3.5. This shows that every projection of \mathcal{H}_μ is trivial and hence we have $\mathcal{H}_\mu = \mathbb{C}1$. \square

It is also well-known that if μ is a non-degenerate measure on a locally compact group G , then every continuous μ -harmonic function on G that vanishes at infinity is constant. We close this section by proving two non-commutative versions of this result.

Theorem 3.7. *Let \mathbb{G} be a locally compact quantum group and let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate. Then we have $\mathcal{H}_\mu \cap \mathcal{K}(L_2(\mathbb{G})) \subseteq \mathbb{C}1$.*

Proof. It follows from the duality between $\mathcal{K}(L_2(\mathbb{G}))$ and $\mathcal{T}(L_2(\mathbb{G}))$, and the fact that $\mathcal{T}(L_2(\mathbb{G}))|_{L_\infty(\mathbb{G})} = L_1(\mathbb{G})$, that for every self-adjoint element $x \in L_\infty(\mathbb{G}) \cap \mathcal{K}(L_2(\mathbb{G}))$, either x or $-x$ attains its norm on $L_1(\mathbb{G})_1^+$. Hence, by Lemma 3.5 we have $x \in \mathbb{C}1$, and since $\mathcal{H}_\mu \cap \mathcal{K}(L_2(\mathbb{G}))$ is generated by its selfadjoint elements, the theorem follows. \square

Theorem 3.8. *Let \mathbb{G} be a locally compact quantum group and let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate. Then we have $\mathcal{H}_\mu \cap C_0(\mathbb{G}) \subseteq \mathbb{C}1$.*

Proof. Suppose that $x \in \mathcal{H}_\mu \cap C_0(\mathbb{G})$ is a self-adjoint element and that $\|x\| = 1$. Then we can find (by substituting x with $-x$, if necessary) a state $\phi \in M(\mathbb{G}) = C_0(\mathbb{G})^*$ such that $\langle x, \phi \rangle = 1$. Now, a similar argument to the proof of Lemma 3.5 shows that $x \in \mathbb{C}1$. Since $\mathcal{H}_\mu \cap C_0(\mathbb{G})$ is generated by its self-adjoint elements, the theorem follows. \square

As a consequence of Theorem 3.8, we obtain the following result.

Corollary 3.9. *Let \mathbb{G} be a non-compact locally compact quantum group and let $\mu \in \mathcal{P}_u(\mathbb{G})$ be non-degenerate. Then the Cesàro sums $\{\frac{1}{n}(\mu + \mu^2 + \cdots + \mu^n)\}$ converge to 0 in the weak* topology.*

Proof. Let $\omega \in M_u(\mathbb{G})$ be an arbitrary weak* cluster point of the Cesàro sums $\{\frac{1}{n}(\mu + \mu^2 + \cdots + \mu^n)\}$ in $M_u(\mathbb{G})$. Then we get $\mu \star_u \omega = \omega$ and thus ω is an idempotent state in $M_u(\mathbb{G})$. For any $x \in C_0(\mathbb{G})$, we have

$$\Phi_\mu(\Phi_\omega(x)) = \Phi_{\mu \star_u \omega}(x) = \Phi_\omega(x),$$

which implies that $\Phi_\omega(x) \in \mathcal{H}_\mu \cap C_0(\mathbb{G})$, and hence, by Theorem 3.9, we have $\Phi_\omega(x) \in \mathbb{C}1$. Since \mathbb{G} is non-compact, this yields that $\Phi_\omega(x) = 0$ for all $x \in C_0(\mathbb{G})$, and therefore it follows from normality of the

map Φ_ω that $\omega = 0$. This shows that zero is the only weak* cluster point of $\{\frac{1}{n}(\mu + \mu^2 + \dots + \mu^n)\}$. Since the unit ball of $M_u(\mathbb{G})$ is weak* compact, we get $\text{weak}^*\text{-}\lim_n \frac{1}{n}(\mu + \mu^2 + \dots + \mu^n) = 0$. \square

Corollary 3.10. *A locally compact quantum group \mathbb{G} is compact if and only if there exists a non-degenerate (and thus faithful) idempotent state $\mu \in \mathcal{P}_u(\mathbb{G})$.*

Finally we remark that all the above results remain valid if we start with a state $\mu \in M(\mathbb{G})$, which is non-degenerate on $C_0(\mathbb{G})$.

4. AMENABILITY OF QUANTUM GROUPS

Our goal in this section is to prove a theorem establishing the equivalence between amenability of a locally compact quantum group \mathbb{G} and the absence of non-trivial harmonic operators on \mathbb{G} (see Theorem 4.2). This answers the quantum group version of a conjecture formulated in the group case by Furstenberg [13], which in the classical setting was answered independently by Kaimanovich–Vershik [24] and Rosenblatt [34].

Let us first recall that a locally compact quantum group \mathbb{G} is *amenable* if there exists a *left invariant mean* on $L_\infty(\mathbb{G})$, i.e., a state $F : L_\infty(\mathbb{G}) \rightarrow \mathbb{C}$ such that $(\iota \otimes F)\Gamma(x) = F(x)1$. Then a standard argument shows that we can find a net of normal states $\{\omega_\alpha\}$ in $L_1(\mathbb{G})$ such that

$$(4.1) \quad \|f \star \omega_\alpha - f(1)\omega_\alpha\| \rightarrow 0$$

for all $f \in L_1(\mathbb{G})$. The following argument is inspired by [24, Theorem 4.3].

Theorem 4.1. *Let \mathbb{G} be a locally compact quantum group such that $L_1(\mathbb{G})$ is separable. Then the following are equivalent:*

- (1) \mathbb{G} is amenable;
- (2) *there exists a state $\mu \in L_1(\mathbb{G})$ such that $\|f \star \mu^n - f(1)\mu^n\| \rightarrow 0$ for every $f \in L_1(\mathbb{G})$, where $\mu^n = \mu \star \dots \star \mu$ is the n -fold convolution of μ .*

Proof. We only need to prove (1) \Rightarrow (2). Let $\{f_i\}_{i \in \mathbb{N}}$ be a dense subset of the unit ball of $L_1(\mathbb{G})$, and let $\{n_k\}$ be an increasing sequence of positive integers such that $(\sum_{i=1}^k \frac{1}{2^i})^{n_k} < \frac{1}{2^k}$. Since \mathbb{G} is amenable, we can apply (4.1) to choose inductively a sequence of states $\{\omega_l\}_{l \in \mathbb{N}}$ in $L_1(\mathbb{G})$ such that

$$\|\omega_{k_1} \star \dots \star \omega_{k_r} \star \omega_l - \omega_l\| < \frac{1}{2^l}$$

for all $1 \leq k_i < l$ with $i = 1, \dots, r \leq n_l$, and such that

$$\|f_s \star \omega_{k_1} \star \dots \star \omega_{k_r} \star \omega_l - f_s(1)\omega_l\| < \frac{1}{2^l}$$

for all $1 \leq s, k_i < l$ with $i = 1, \dots, r \leq n_l$. Define the normal state $\mu = \sum_{l=1}^\infty \frac{1}{2^l} \omega_l \in L_1(\mathbb{G})$. Now given any f in the unit ball of $L_1(\mathbb{G})$ and $\epsilon > 0$, we can choose $j \in \mathbb{N}$ such that $\|f - f_j\| < \epsilon$ and $\frac{1}{2^j} < \epsilon$. To simplify our notation, we fix $p = n_j$ and write $t_i = \frac{1}{2^i}$. Then we get

$$\|f \star \mu^p - f(1)\mu^p\| \leq \|f \star \mu^p - f_j \star \mu^p\| + \|f_j \star \mu^p - f_j(1)\mu^p\| + \|f_j(1)\mu^p - f(1)\mu^p\|$$

$$< 2\epsilon + \|f_j \star \mu^p - f_j(1)\mu^p\|.$$

Now we split the term $\|f_j \star \mu^p - f_j(1)\mu^p\|$ as follows:

$$\begin{aligned} \|f_j \star \mu^p - f_j(1)\mu^p\| &= \left\| \sum_{\max k_i \leq j} t_{k_1} \dots t_{k_p} f_j \star \omega_{k_1} \star \dots \star \omega_{k_p} + \sum_{\max k_i > j} t_{k_1} \dots t_{k_p} f_j \star \omega_{k_1} \star \dots \star \omega_{k_p} \right. \\ &\quad \left. - \sum_{\max k_i \leq j} f_j(1) t_{k_1} \dots t_{k_p} \omega_{k_1} \star \dots \star \omega_{k_p} - \sum_{\max k_i > j} f_j(1) t_{k_1} \dots t_{k_p} \omega_{k_1} \star \dots \star \omega_{k_p} \right\| \\ &\leq \sum_{\max k_i \leq j} 2\|f_j\| t_{k_1} \dots t_{k_p} + \sum_{\max k_i > j} t_{k_1} \dots t_{k_p} \|f_j \star \omega_{k_1} \star \dots \star \omega_{k_p} - f_j(1)\omega_{k_1} \star \dots \star \omega_{k_p}\| \\ &\leq 2\epsilon + \sum_{\max k_i > j} t_{k_1} \dots t_{k_p} \|f_j \star \omega_{k_1} \star \dots \star \omega_{k_p} - f_j(1)\omega_{k_1} \star \dots \star \omega_{k_p}\|. \end{aligned}$$

Now consider one of the terms, $\omega_{k_1} \star \dots \star \omega_{k_p}$, in the last sum above and let k_j be the smallest index such that $k_j > j$. Let $\mu_1 = \omega_{k_1} \star \dots \star \omega_{k_{j-1}}$ and $\mu_2 = \omega_{k_{j+1}} \star \dots \star \omega_{k_p}$. Then we have

$$\begin{aligned} \|f_j \star \omega_{k_1} \star \dots \star \omega_{k_p} - f_j(1)\omega_{k_1} \star \dots \star \omega_{k_p}\| &= \|f_j \star \mu_1 \star \omega_{k_j} \star \mu_2 - f_j(1)\mu_1 \star \omega_{k_j} \star \mu_2\| \\ &\leq \|f_j \star \mu_1 \star \omega_{k_j} - f_j(1)\mu_1 \star \omega_{k_j}\| + \|f_j(1)\mu_1 \star \omega_{k_j} - f_j(1)\omega_{k_j}\| < 2\epsilon, \end{aligned}$$

where the last inequality follows from the construction of $\{\omega_l\}$. This implies

$$\sum_{\max k_i > j} t_{k_1} \dots t_{k_p} \|f_j \star \omega_{k_1} \star \dots \star \omega_{k_p} - f_j(1)\omega_{k_1} \star \dots \star \omega_{k_p}\| < 2\epsilon.$$

Hence we have $\|f \star \mu^p - f(1)\mu^p\| < 6\epsilon$. Since $\|\mu\| = 1$, we have

$$\|f \star \mu^{p+l} - f(1)\mu^{p+l}\| \leq \|f \star \mu^p - f(1)\mu^p\| < 6\epsilon$$

for all $l \in \mathbb{N}$. This implies that $\|f \star \mu^n - f(1)\mu^n\| \rightarrow 0$. \square

Theorem 4.2. *Let \mathbb{G} be a locally compact quantum group such that $L_1(\mathbb{G})$ is separable. Then the following are equivalent:*

- (1) \mathbb{G} is amenable;
- (2) there exists a state $\mu \in M(\mathbb{G})$ such that $\mathcal{H}_\mu = \mathbb{C}1$.

Proof. Recall that we denote by Φ_μ the Markov operator $x \mapsto \mu \star x$ ($x \in L_\infty(\mathbb{G})$). Let us first assume that \mathbb{G} is amenable. It follows from Theorem 4.1 that there exists $\mu \in L_1(\mathbb{G})$ such that $\|f \star \mu^n - f(1)\mu^n\| \rightarrow 0$ for every $f \in L_1(\mathbb{G})$. Given any $x \in \mathcal{H}_\mu$ and $n \in \mathbb{N}$, we have $\Phi_{\mu^n}(x) = \Phi_\mu^n(x) = x$. It follows that for every $f \in L_1(\mathbb{G})$, we have

$$\langle f, x - \mu^n(x)1 \rangle = \langle f, \Phi_{\mu^n}(x) - \mu^n(x)1 \rangle = \langle f \star \mu^n - f(1)\mu^n, x \rangle \rightarrow 0.$$

This implies that $\mu^n(x)1 \rightarrow x$ in the weak* topology, and thus we get $x \in \mathbb{C}1$. This shows that $\mathcal{H}_\mu = \mathbb{C}1$.

On the other hand, let us suppose that we have a state $\mu \in M(\mathbb{G})$ such that $\mathcal{H}_\mu = \mathbb{C}1$. We choose a normal state $f \in L_1(\mathbb{G})$. Then for each $n \in \mathbb{N}$, we get a normal state $\mu_n = \frac{1}{n} \sum_{k=1}^n \mu^k \star f \in L_1(\mathbb{G})$. Let $F = \lim_{\mathcal{U}} \mu_n \in L_\infty(\mathbb{G})^*$ be the weak* limit of $\{\mu_n\}$ with respect to a free ultrafilter \mathcal{U} on \mathbb{N} . Then F is

a state on $L_\infty(\mathbb{G})$. We claim that $(\iota \otimes F)\Gamma(x) \in \mathcal{H}_\mu = \mathbb{C}1$ for all $x \in L_\infty(\mathbb{G})$. To see this, we notice that the Markov operator Φ_μ satisfies

$$\begin{aligned} \langle \Phi_\mu((\iota \otimes F)\Gamma(x)), g \rangle &= \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \langle (\iota \otimes \mu^k \star f)\Gamma(x), g \star \mu \rangle \\ &= \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \langle (g \star \mu) \star (\mu^k \star f), x \rangle = \langle (\iota \otimes F)\Gamma(x), g \rangle \end{aligned}$$

for all $g \in L_1(\mathbb{G})$. This shows that $(\iota \otimes F)\Gamma(x)$ is an element in $\mathcal{H}_\mu = \mathbb{C}1$.

We define $F' \in L_\infty(\mathbb{G})^*$ such that $F'(x)1 = (\iota \otimes F)\Gamma(x)$. Applying μ to both sides of the latter, we obtain

$$F'(x) = F'(x)\mu(1) = \mu(F'(x)1) = \mu((\iota \otimes F)\Gamma(x)) = F(x)$$

for all $x \in L_\infty(\mathbb{G})$. Therefore, we get that $(\iota \otimes F')\Gamma(x) = F'(x)1$ and thus \mathbb{G} is amenable. \square

Remark 4.3. *Using the same proof as that given in Theorem 4.2, we can show that a locally compact quantum group \mathbb{G} is amenable if and only if there exists a state $\omega \in \mathcal{P}_u(\mathbb{G})$ such that $\mathcal{H}_\omega = \mathbb{C}1$, if and only if there exists a normal state $f \in L_1(\mathbb{G})$ such that $\mathcal{H}_f = \mathbb{C}1$.*

5. THE COMPACT QUANTUM GROUP CASE

In this section, we consider compact quantum groups \mathbb{G} . Our goal is to prove (Theorem 5.3 below) a compact quantum group analogue of the Choquet–Deny theorem.

Since \mathbb{G} is compact, its reduced quantum group C^* -algebra $C_0(\mathbb{G})$ and its universal quantum group C^* -algebra $C_u(\mathbb{G})$ are unital Hopf C^* -algebras with the comultiplication $\Gamma : C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$ and the universal comultiplication $\Gamma_u : C_u(\mathbb{G}) \rightarrow C_u(\mathbb{G}) \otimes C_u(\mathbb{G})$, respectively. Also, in this case the C^* -algebra $C_0(\mathbb{G})$ is equal to the multiplier algebra $C(\mathbb{G}) = M(C_0(\mathbb{G}))$. If ϕ is an idempotent state in $\mathcal{P}_u(\mathbb{G})$, i.e., $\phi \star_u \phi = \phi$, it was shown in [11, Theorem 4.1] that $\tilde{\mathcal{H}}_\phi = \{x_u \in C_u(\mathbb{G}) : \tilde{\Phi}_\phi(x_u) = x_u\}$ is a C^* -subalgebra of $C_u(\mathbb{G})$, where

$$(5.1) \quad \tilde{\Phi}_\phi(x_u) = (\iota \otimes \phi)\Gamma_u(x_u) = \phi \star_u x_u.$$

Using this fact, we can prove that for the corresponding Markov operator $\Phi_\phi = (\mathfrak{m}_\phi^r)^*$ on $L_\infty(\mathbb{G})$, the Poisson boundary \mathcal{H}_ϕ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$. Let us first establish a lemma.

Lemma 5.1. *Let \mathbb{G} be a compact quantum group and let $\phi \in \mathcal{P}_u(\mathbb{G})$ be an idempotent state. Then we have*

$$(5.2) \quad \Phi_\phi(\Phi_\phi(x)\Phi_\phi(y)) = \Phi_\phi(x)\Phi_\phi(y)$$

for all $x, y \in L_\infty(\mathbb{G})$. Moreover, the Poisson boundary \mathcal{H}_ϕ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$.

Proof. We first note that as an immediate consequence of (3.3) and (5.1), we get

$$(5.3) \quad \Phi_\phi(\pi(x_u)) = \pi(\tilde{\Phi}_\phi(x_u))$$

for all $x_u \in C_u(\mathbb{G})$. Now for any $x, y \in C(\mathbb{G})$, we can find $x_u, y_u \in C_u(\mathbb{G})$ such that $x = \pi(x_u)$ and $y = \pi(y_u)$, and thus we obtain

$$\begin{aligned} \Phi_\phi(\Phi_\phi(x)\Phi_\phi(y)) &= \Phi_\phi(\Phi_\phi(\pi(x_u))\Phi_\phi(\pi(y_u))) = \Phi_\phi(\pi(\tilde{\Phi}_\phi(x_u))\pi(\tilde{\Phi}_\phi(y_u))) \\ &= \Phi_\phi(\pi(\tilde{\Phi}_\phi(x_u)\tilde{\Phi}_\phi(y_u))) = \pi(\tilde{\Phi}_\phi(\tilde{\Phi}_\phi(x_u)\tilde{\Phi}_\phi(y_u))) \stackrel{(*)}{=} \pi(\tilde{\Phi}_\phi(x_u)\tilde{\Phi}_\phi(y_u)) \\ &= \pi(\tilde{\Phi}_\phi(x_u))\pi(\tilde{\Phi}_\phi(y_u)) = \Phi_\phi(\pi(x_u))\Phi_\phi(\pi(y_u)) = \Phi_\phi(x)\Phi_\phi(y) \end{aligned}$$

where we used [11, Theorem 4.1] in $(*)$. It is known from the Kaplansky density theorem that the closed unit ball of $C(\mathbb{G})$ is weak* dense in the closed unit ball of $L_\infty(\mathbb{G})$. Then for any contractive $x \in \mathcal{H}_\phi \subseteq L_\infty(\mathbb{G})$, there exists a net of contractive elements $x_i \in C(\mathbb{G})$ such that $x_i \rightarrow x$ in the weak* topology. Since Φ_ϕ is weak* continuous, we get $\Phi_\phi(x_i) \rightarrow \Phi_\phi(x) = x$ in the weak* topology. Similarly, for any $y \in \mathcal{H}_\phi$, we can find a net of elements $y_j \in C(\mathbb{G})$ such that $\Phi_\phi(y_j) \rightarrow y$ in the weak* topology. Then we get the following iterated weak* limit

$$\Phi_\phi(xy) = \lim_i \lim_j \Phi_\phi(\Phi_\phi(x_i)\Phi_\phi(y_j)) = \lim_i \lim_j \Phi_\phi(x_i)\Phi_\phi(y_j) = xy.$$

This shows that the Choi–Effros product on \mathcal{H}_ϕ coincides with the product on $L_\infty(\mathbb{G})$. Therefore, \mathcal{H}_ϕ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$. \square

Theorem 5.2. *Let \mathbb{G} be a compact quantum group and let ω be in $\mathcal{P}_u(\mathbb{G})$. Then there exists an idempotent state $\phi \in \mathcal{P}_u(\mathbb{G})$ such that $\mathcal{H}_\omega = \mathcal{H}_\phi$ is a von Neumann subalgebra of $L_\infty(\mathbb{G})$.*

Proof. Consider the Cesàro sums $\omega_n = \frac{1}{n}(\omega + \dots + \omega^n)$, $n \in \mathbb{N}$, and take the weak* limit $\phi = \lim_{\mathcal{U}} \omega_n$ with respect to a free ultrafilter \mathcal{U} on \mathbb{N} . Then ϕ is an idempotent state in $\mathcal{P}_u(\mathbb{G})$ such that $\phi \star_u \omega = \phi = \omega \star_u \phi$. Clearly, $\mathcal{H}_\phi \subseteq \mathcal{H}_\omega$ since for any $x \in \mathcal{H}_\phi$, we have

$$\Phi_\omega(x) = \Phi_\omega(\Phi_\phi(x)) = \Phi_{\omega \star_u \phi}(x) = \Phi_\phi(x) = x.$$

To prove the converse inclusion, let us first suppose that $x \in C(\mathbb{G}) \cap \mathcal{H}_\omega$. Since \mathbb{G} is compact, we have $C(\mathbb{G}) = C_0(\mathbb{G})$ and so there exists $x_u \in C_u(\mathbb{G})$ such that $x = \pi(x_u)$. Then for any $f \in L_1(\mathbb{G})$, we have

$$\begin{aligned} \langle \Phi_\phi(x), f \rangle &= \langle x, f \star \phi \rangle = \langle x_u, \pi^*(f) \star_u \phi \rangle = \langle x_u \star_u \pi^*(f), \phi \rangle = \lim_{\mathcal{U}} \langle x_u \star_u \pi^*(f), \omega_n \rangle \\ &= \lim_{\mathcal{U}} \langle x_u, \pi^*(f) \star_u \omega_n \rangle = \lim_{\mathcal{U}} \langle \Phi_{\omega_n}(x), f \rangle = \langle x, f \rangle. \end{aligned}$$

This shows that $x \in \mathcal{H}_\phi$. Hence, $\mathcal{H}_\omega \cap C(\mathbb{G}) \subseteq \mathcal{H}_\phi$.

Now let $x \in \mathcal{H}_\omega$. Then for any $f \in L_1(\mathbb{G})$ we have $x \star f \in LUC(\mathbb{G}) = C(\mathbb{G})$. We also have $x \star f \in \mathcal{H}_\omega$ since $\Phi_\omega(x \star f) = \Phi_\omega(x) \star f = x \star f$. Therefore we have $x \star f \in C(\mathbb{G}) \cap \mathcal{H}_\omega \subseteq \mathcal{H}_\phi$ for all $f \in L_1(\mathbb{G})$. From this we conclude that

$$\langle \Phi_\phi(x), f \star g \rangle = \langle x, f \star g \star \phi \rangle = \langle x \star f, g \star \phi \rangle = \langle \Phi_\phi(x \star f), g \rangle = \langle x \star f, g \rangle = \langle x, f \star g \rangle$$

for all $f, g \in L_1(\mathbb{G})$. Since $L_1(\mathbb{G}) = \langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle$, we obtain $\Phi_\phi(x) = x$. Hence, $\mathcal{H}_\omega \subseteq \mathcal{H}_\phi$. \square

Now, as a corollary to Theorems 3.8 and 5.2, we have the following compact quantum group analogue of the Choquet–Deny theorem. A special case of this result was proved by Franz and Skalski [12] where ω was assumed to be faithful.

Theorem 5.3. *Let \mathbb{G} be a compact quantum group and let $\omega \in \mathcal{P}_u(\mathbb{G})$ be a non-degenerate state. Then we have $\mathcal{H}_\omega = \mathbb{C}1$.*

The following result shows that for compact quantum groups, the faithful Haar state arises from any non-degenerate state.

Corollary 5.4. *Let \mathbb{G} be a compact quantum group and let μ be a (not necessarily faithful) non-degenerate state in $M(\mathbb{G})$. For any free ultrafilter \mathcal{U} on \mathbb{N} , the weak* limit $\phi = \lim_{\mathcal{U}} \frac{1}{n}(\mu + \cdots + \mu^n) \in M(\mathbb{G})$ is the faithful Haar state of \mathbb{G} .*

Proof. Since μ is a non-degenerate state in $M(\mathbb{G})$, it follows from Theorem 5.3 that $\mathcal{H}_\mu = \mathbb{C}1$. Let ϕ be the idempotent state given by the weak* limit $\phi = \lim_{\mathcal{U}} \frac{1}{n}(\mu + \cdots + \mu^n)$. Then for any $x \in C(\mathbb{G})$, we have

$$(\iota \otimes \phi)\Gamma(x) = \Phi_\phi(x) \in \mathcal{H}_\mu = \mathbb{C}1.$$

So we can write $(\iota \otimes \phi)\Gamma(x) = F'(x)1$. Since $\mu \star \phi = \phi$, we can apply the same argument as we have shown in the proof of Theorem 4.2 and get $F'(x) = \phi(x)$. Therefore, ϕ must be the Haar state of \mathbb{G} . \square

Applying Corollary 3.10, we obtain the following interesting result of Fima [10, Theorem 8].

Corollary 5.5. *Let \mathbb{G} be a locally compact quantum group such that $L_\infty(\mathbb{G})$ is a finite factor. Then \mathbb{G} is a compact Kac algebra.*

Proof. Let $\tau \in L_1(\mathbb{G})$ be the unique faithful trace. Then the uniqueness of τ implies that $\tau^2 = \tau$, and hence, it follows from Corollary 3.10 that \mathbb{G} is compact. Moreover, Corollary 5.5 implies that the trace τ is the Haar state of \mathbb{G} , and so \mathbb{G} is a Kac algebra. \square

6. EXAMPLES

It is often highly non-trivial to concretely identify Poisson boundaries associated to a given locally compact quantum group. The situation in the classical setting is of course much easier. The structure of Poisson boundaries has been studied in detail for locally compact groups in many interesting cases.

In this section we will establish a bridge between the classical and the quantum setting, through a concrete formula (6.4), which then allows us to link the Poisson boundaries in these two settings. In particular, we apply our machinery to the case of Woronowicz' twisted $SU_q(2)$ and show that the Poisson boundary associated to a specific state on this compact quantum group can be identified with the Podleś sphere.

Throughout this section, \mathbb{G} denotes a co-amenable locally compact quantum group. Let us recall that in this case we have

$$M(\mathbb{G}) = M_u(\mathbb{G}) = C_{cb}^r(L_1(\mathbb{G})).$$

Proposition 6.1. *Let \mathbb{G} be a co-amenable locally compact quantum group. If μ is a state in $M(\mathbb{G}) = M_u(\mathbb{G})$, then the closed unit ball of $\mathcal{H}_\mu \cap LUC(\mathbb{G})$ is weak* dense in the closed unit ball of \mathcal{H}_μ .*

Proof. Let $y \in \mathcal{H}_\mu$ with $\|y\| \leq 1$ and let $\{f_\alpha\} \subseteq L_1(\mathbb{G})$ be a contractive approximate identity. Then we have

$$\Phi_\mu((f_\alpha \otimes \iota)\Gamma(y)) = (f_\alpha \otimes \iota)(\iota \otimes \Phi_\mu)\Gamma(y) = (f_\alpha \otimes \iota)\Gamma(\Phi_\mu(y)) = (f_\alpha \otimes \iota)\Gamma(y),$$

which implies that $y \star f_\alpha = (f_\alpha \otimes \iota)\Gamma(y) \in \mathcal{H}_\mu \cap LUC(\mathbb{G})$. Since $\{f_\alpha\}$ is a contractive approximate identity, we have $\|y \star f_\alpha\| \leq 1$ and $(f_\alpha \otimes \iota)\Gamma(y) \rightarrow y$ in the weak* topology. This completes the proof. \square

It was shown in Kalantar's thesis [25, Chapter 3] that if \mathbb{G} is a co-amenable locally compact quantum group, the *spectrum*

$$\tilde{\mathbb{G}} = sp(C_0(\mathbb{G})) = \{\phi : C_0(\mathbb{G}) \rightarrow \mathbb{C} \mid \phi \text{ is a non-zero } *-homomorphism\}$$

of $C_0(\mathbb{G})$ equipped with the convolution product and the weak* topology from $M(\mathbb{G})$ is a locally compact group. We let

$$\wedge : x \in C_0(\mathbb{G}) \rightarrow \hat{x} \in C_0(\mathbb{G})^{**}$$

be the canonical second dual inclusion and let

$$(6.1) \quad P : x \in C_0(\mathbb{G}) \rightarrow \hat{x} \mid_{\tilde{\mathbb{G}}} \in C_0(\tilde{\mathbb{G}})$$

be the Gelfand transformation given by $P(x)(\phi) = \phi(x)$ for all $\phi \in \tilde{\mathbb{G}}$.

Proposition 6.2. *Let \mathbb{G} be a co-amenable locally compact quantum group. The map P defined in (6.1) is a *-homomorphism from $C_0(\mathbb{G})$ onto $C_0(\tilde{\mathbb{G}})$, and thus has a unique strictly continuous unital *-homomorphism extension from the C^* -multiplier algebra $C(\mathbb{G}) = M(C_0(\mathbb{G}))$ onto the C^* -multiplier algebra $C(\tilde{\mathbb{G}}) = M(C_0(\tilde{\mathbb{G}}))$.*

Proof. It is easy to see that P is a *-homomorphism from $C_0(\mathbb{G})$ into $C_0(\tilde{\mathbb{G}})$. Then the range space of P is a C^* -subalgebra of $C_0(\tilde{\mathbb{G}})$ and it separates points in $\tilde{\mathbb{G}}$. Therefore, by the generalized Stone-Weierstrass theorem, we have $P(C_0(\mathbb{G})) = C_0(\tilde{\mathbb{G}})$. Therefore, P has a unique strictly continuous unital *-homomorphism extension, mapping the C^* -multiplier algebra $C(\mathbb{G}) = M(C_0(\mathbb{G}))$ onto the C^* -multiplier algebra $C(\tilde{\mathbb{G}}) = M(C_0(\tilde{\mathbb{G}}))$ (cf. Lance [29]). \square

Since the comultiplication $\Gamma : C_0(\mathbb{G}) \rightarrow M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ and the *-homomorphisms

$$\iota \otimes P : C_0(\mathbb{G}) \otimes C_0(\mathbb{G}) \rightarrow C_0(\mathbb{G}) \otimes C_0(\tilde{\mathbb{G}}) \text{ and } P \otimes P : C_0(\mathbb{G}) \otimes C_0(\mathbb{G}) \rightarrow C_0(\tilde{\mathbb{G}}) \otimes C_0(\tilde{\mathbb{G}})$$

have unique strictly continuous extensions to their C^* -multiplier algebras, we can consider their compositions with the comultiplication Γ and obtain the following result.

Proposition 6.3. *Let \mathbb{G} be a co-amenable locally compact quantum group and let $P : C(\mathbb{G}) \rightarrow C(\tilde{\mathbb{G}})$ be the strictly continuous unital *-homomorphism defined above.*

(1) *If we let Γ_a denote the comultiplication on $C_0(\tilde{\mathbb{G}})$, we have*

$$(6.2) \quad (P \otimes P) \circ \Gamma = \Gamma_a \circ P.$$

(2) *The induced map $(\iota \otimes P) \circ \Gamma$ is an injective *-homomorphisms from $C(\mathbb{G})$ into $M(C_0(\mathbb{G}) \otimes C_0(\tilde{\mathbb{G}}))$.*

Proof. The first part follows from straightforward calculations. For the second part, let ε be the unital element in $M(\mathbb{G})$. So, we have $(\iota \otimes \varepsilon)\Gamma(x) = x$ for all $x \in C_0(\mathbb{G})$, and thus for all $x \in C(\mathbb{G})$. Moreover, ε is a non-zero *-homomorphism, and thus is an element, which is denoted by e , in $\tilde{\mathbb{G}}$. Since the multiplication of the group $\tilde{\mathbb{G}}$ is induced from the multiplication of $M(\mathbb{G})$, e is just the unital element of $\tilde{\mathbb{G}}$. Moreover, for any $x \in C_0(\mathbb{G})$, we have

$$\varepsilon(x) = \hat{x}(\varepsilon) = P(x)(e) = e(P(x)).$$

This implies that $\varepsilon = e \circ P$. Now, if we are given $x \in C(\mathbb{G})$ such that $(\iota \otimes P)\Gamma(x) = 0$, then we have

$$x = (\iota \otimes \varepsilon)\Gamma(x) = (\iota \otimes e)(\iota \otimes P)\Gamma(x) = 0.$$

So $(\iota \otimes P) \circ \Gamma$ is injective. □

Since P is a *-homomorphism from $C_0(\mathbb{G})$ onto $C_0(\tilde{\mathbb{G}})$ satisfying (6.2), its adjoint map P^* defines a completely isometric and Banach algebraic homomorphism

$$(6.3) \quad P^* : M(\tilde{\mathbb{G}}) \ni \mu \mapsto \mu_{\mathbb{G}} = \mu \circ P \in M(\mathbb{G}).$$

Therefore, we can identify $M(\tilde{\mathbb{G}})$ with a Banach subalgebra of $M(\mathbb{G})$.

Theorem 6.4. *Let \mathbb{G} be a co-amenable locally compact quantum group and let μ be a probability measure in $M(\tilde{\mathbb{G}})$. Then we have*

$$(6.4) \quad \mathcal{H}_{\mu_{\mathbb{G}}} = \overline{\{x \in C(\mathbb{G}) : (\iota \otimes P)\Gamma(x) \in L_{\infty}(\mathbb{G}) \otimes \mathcal{H}_{\mu}\}}^{weak*}.$$

Proof. We prove that

$$\mathcal{H}_{\mu_{\mathbb{G}}} \cap C(\mathbb{G}) = \{x \in C(\mathbb{G}) : (\iota \otimes P)\Gamma(x) \in L_{\infty}(\mathbb{G}) \otimes \mathcal{H}_{\mu_{\mathbb{G}}}\}.$$

from which the result follows by Proposition 6.1.

Given any $x \in C(\mathbb{G})$, we have by (6.2) that

$$\Phi_{\mu} \circ P(x) = (\iota \otimes \mu)\Gamma_a(P(x)) = (\iota \otimes \mu)(P \otimes P)(\Gamma(x)) = P((\iota \otimes \mu_{\mathbb{G}})\Gamma(x)) = P \circ \Phi_{\mu_{\mathbb{G}}}(x).$$

Therefore, if $x \in \mathcal{H}_{\mu_{\mathbb{G}}} \cap C(\mathbb{G})$, then we get $P(x) \in \mathcal{H}_{\mu} \cap C(\tilde{\mathbb{G}})$ and

$$(\iota \otimes \Phi_{\mu})(\iota \otimes P)\Gamma(x) = (\iota \otimes P)(\iota \otimes \Phi_{\mu_{\mathbb{G}}})\Gamma(x) = (\iota \otimes P)\Gamma(\Phi_{\mu_{\mathbb{G}}}(x)) = (\iota \otimes P)\Gamma(x).$$

Hence, $(\iota \otimes P)\Gamma(x) \in L_{\infty}(\mathbb{G}) \otimes \mathcal{H}_{\mu}$ (cf. section 7).

On the other hand, assume that $x \in C(\mathbb{G})$ is such that $(\iota \otimes P)\Gamma(x) \in L_\infty(\mathbb{G}) \overline{\otimes} \mathcal{H}_\mu$. Then we have

$$(\iota \otimes P)\Gamma(\Phi_{\mu_{\mathbb{G}}}(x)) = (\iota \otimes P)(\iota \otimes \Phi_{\mu_{\mathbb{G}}})\Gamma(x) = (\iota \otimes \Phi_\mu)(\iota \otimes P)\Gamma(x) = (\iota \otimes P)\Gamma(x).$$

Hence, by Proposition 6.3, we obtain $\Phi_{\mu_{\mathbb{G}}}(x) = x$. \square

In the following, we consider Woronowicz's twisted $SU_q(2)$ quantum group for $q \in (-1, 1)$ and $q \neq 0$ (cf. [40]). It is known that $SU_q(2)$ is a co-amenable compact quantum group with the quantum group C^* -algebra $C(SU_q(2)) = C_u(SU_q(2))$ generated by two operators u and v such that $U = \begin{bmatrix} u & -qv^* \\ v & u^* \end{bmatrix}$ is a unitary matrix in $M_2(C(SU_q(2)))$.

It was shown in [25, Theorem 3.4.3] that $\tilde{\mathbb{G}}$ is actually homeomorphic to the unit circle group \mathbb{T} . Indeed, if $f \in \tilde{\mathbb{G}}$ is a non-zero $*$ -homomorphism on $C(SU_q(2))$, then

$$\begin{bmatrix} f(u) & f(-qv^*) \\ f(v) & f(u^*) \end{bmatrix} = \begin{bmatrix} f(u) & -q\overline{f(v)} \\ f(v) & \overline{f(u)} \end{bmatrix}$$

is a unitary matrix in $M_2(\mathbb{C})$. This implies that

$$|f(u)|^2 + |f(v)|^2 = 1 \text{ and } |f(u)|^2 + q^2|f(v)|^2 = 1.$$

Since $0 < |q| < 1$, we must have $f(v) = 0$ and $|f(u)| = 1$. Then we get a map

$$\gamma : \widetilde{SU_q(2)} \ni f \mapsto \begin{bmatrix} f(u) & 0 \\ 0 & \overline{f(u)} \end{bmatrix}$$

which gives a map from $\widetilde{SU_q(2)}$ into the unit circle \mathbb{T} . Since $C(SU_q(2))$ is the universal C^* -algebra generated by u and v , it is easy to see that γ defines a homeomorphism from $\widetilde{SU_q(2)}$ onto \mathbb{T} . Moreover, since $\widetilde{SU_q(2)}$ is a compact group, and $\Gamma(u) = u \otimes u$ (see [40, Theorem 1.4]), γ is a group homeomorphism from $\widetilde{SU_q(2)}$ onto \mathbb{T} . Therefore, we can identify the spectrum $\widetilde{SU_q(2)}$ with \mathbb{T} .

In view of the above discussion, the $*$ -homomorphism P defined in (6.1) can be identified with a map

$$P_\gamma : x \in C(SU_q(2)) \rightarrow \hat{x} \circ \gamma^{-1} \in C(\mathbb{T})$$

such that $P_\gamma(u) = \iota_{\mathbb{T}}$ and $P_\gamma(v) = 0$, where $\iota_{\mathbb{T}} : \mathbb{T} \rightarrow \mathbb{T}$ is the identity function $z \mapsto z$ on \mathbb{T} . Now let

$$(6.5) \quad C(SU_q(2) \setminus \mathbb{T}) = \{x \in C(SU_q(2)) : (\iota \otimes P_\gamma) \circ \Gamma(x) = x \otimes 1\}.$$

Then $C(SU_q(2) \setminus \mathbb{T})$ is a C^* -subalgebra of $C(SU_q(2))$, and one can show that $(C(SU_q(2) \setminus \mathbb{T}), \Gamma|_{C(SU_q(2) \setminus \mathbb{T})})$ is one of the Podles' quantum spheres (see [33] for the details). We also call the von Neumann algebra generated by $C(SU_q(2) \setminus \mathbb{T})$ in $L_\infty(SU_q(2))$ a *quantum sphere* and will denote it by $L_\infty(SU_q(2) \setminus \mathbb{T})$. In the next theorem, we show that the quantum sphere $L_\infty(SU_q(2) \setminus \mathbb{T})$ is a concrete realization of the Poisson boundary of Markov operators associated with non-degenerate measures in $M(\mathbb{T})$.

Theorem 6.5. *Let $\mu \in M(\mathbb{T})$ be a non-degenerate measure. Then we have*

$$(6.6) \quad \mathcal{H}_{\mu_{SU_q(2)}} = L_\infty(SU_q(2) \setminus \mathbb{T}).$$

Proof. It follows from (6.3) and (6.5) that if $x \in C(SU_q(2) \setminus \mathbb{T})$, then we have

$$\Phi_{\mu_{SU_q(2)}}(x) = (\iota \otimes \mu)(\iota \otimes P_\gamma)\Gamma(x) = (\iota \otimes \mu)(x \otimes 1) = x.$$

Hence we see that $L_\infty(SU_q(2) \setminus \mathbb{T}) \subseteq \mathcal{H}_{\mu_{SU_q(2)}}$. For the converse inclusion, by Theorem 6.4 it is enough to show that any $x \in C(SU_q(2))$ with

$$(\iota \otimes P_\gamma)\Gamma(x) \in L_\infty(\mathbb{G}) \overline{\otimes} \mathcal{H}_\mu$$

lies in $C(SU_q(2) \setminus \mathbb{T})$. Since $\mu \in M(\mathbb{T})$ is non-degenerate, $\mathcal{H}_\mu = \mathbb{C}1$, hence $(\iota \otimes P_\gamma)\Gamma(x) \in L_\infty(\mathbb{G}) \overline{\otimes} \mathbb{C}1$. So, there exists $y \in L_\infty(\mathbb{G})$ such that $(\iota \otimes P_\gamma)\Gamma(x) = y \otimes 1$. This implies that

$$x = (\iota \otimes \varepsilon)\Gamma(x) = (\iota \otimes e)(\iota \otimes P)\Gamma(x) = (\iota \otimes e)(y \otimes 1) = y,$$

which yields that $x \in C(SU_q(2) \setminus \mathbb{T})$. \square

7. CONCRETE REALIZATION OF THE POISSON BOUNDARY IN $\mathcal{B}(L_2(\mathbb{G}))$

In this section, \mathbb{G} denotes a general locally compact quantum group. Let $\mu \in \mathcal{P}_u(\mathbb{G})$. It follows from [23, Theorem 4.5] that the Markov operator Φ_μ has a unique weak* continuous (unital completely positive) extension $\Theta(\mu)$ to $\mathcal{B}(L_2(\mathbb{G}))$ such that

$$(7.1) \quad \tilde{\Gamma} \circ \Theta(\mu) = (\iota \otimes \Phi_\mu) \circ \tilde{\Gamma}.$$

Proposition 7.1. *Let $\mu \in \mathcal{P}_u(\mathbb{G})$. Then $\Theta(\mu)$ is a faithful Markov operator on $\mathcal{B}(L_2(\mathbb{G}))$.*

Proof. Since Φ_μ is faithful on $L_\infty(\mathbb{G})$ (Lemma 3.4), $\iota \otimes \Phi_\mu$ is faithful on $\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} L_\infty(\mathbb{G})$. Therefore, $\Theta(\mu)$ is faithful on $\mathcal{B}(L_2(\mathbb{G}))$ by (7.1). \square

Similarly to (3.1) we obtain a projection $\mathcal{E}_{\Theta(\mu)}$ of norm one on $\mathcal{B}(L_2(\mathbb{G}))$ and a von Neumann algebra product on $\mathcal{H}^{\Theta(\mu)}$. We denote this von Neumann algebra by $\mathcal{H}_{\Theta(\mu)}$. The main result of this section is to show that there is a left \mathbb{G} action on \mathcal{H}_μ such that $\mathcal{H}_{\Theta(\mu)}$ is *-isomorphic to the von Neumann algebra crossed product of \mathcal{H}_μ by \mathbb{G} .

For this purpose, we need to recall the Fubini product for weak* closed operator spaces on Hilbert spaces. Suppose that V and W are weak* closed subspaces of $\mathcal{B}(H)$ and $\mathcal{B}(K)$, respectively. We define the *Fubini product* of V and W to be the space

$$V \overline{\otimes}_{\mathcal{F}} W = \{X \in \mathcal{B}(H) \overline{\otimes} \mathcal{B}(K) : (\omega \otimes \iota)(X) \in W \text{ and } (\iota \otimes \phi)(X) \in V \text{ for all } \omega \in \mathcal{B}(H)_*, \phi \in \mathcal{B}(K)_*\}.$$

In this case, $V_* = \mathcal{B}(H)_*/V_\perp$ and $W_* = \mathcal{B}(K)_*/W_\perp$ are operator preduals of V and W , respectively. It is known from [35, Proposition 3.3] (see also [7] and [8, §7.2]) that the Fubini product $V \overline{\otimes}_{\mathcal{F}} W$ is a

weak* closed subspace of $\mathcal{B}(H \otimes K)$ such that we have the weak* homeomorphic completely isometric isomorphism

$$(7.2) \quad V \overline{\otimes}_{\mathcal{F}} W = (V_* \hat{\otimes} W_*)^*,$$

where $V_* \hat{\otimes} W_*$ is the operator space projective tensor product of V_* and W_* . In particular, if M and N are von Neumann algebras, the Fubini product coincides with the von Neumann algebra tensor product, i.e., we have

$$M \overline{\otimes}_{\mathcal{F}} N = M \overline{\otimes} N.$$

It is also known from operator space theory that there is a canonical completely isometrically identification

$$(7.3) \quad (V_* \hat{\otimes} W_*)^* = \mathcal{CB}(V_*, W)$$

given by the left slice maps. Now, if W_1 and W_2 are dual operator spaces and $\Psi : W_1 \rightarrow W_2$ is a (not necessarily weak* continuous) completely bounded map, we can apply (7.2) and (7.3) to obtain a completely bounded map

$$\iota \otimes \Psi : V \overline{\otimes}_{\mathcal{F}} W_1 \rightarrow V \overline{\otimes}_{\mathcal{F}} W_2$$

such that

$$(\omega \otimes \iota)(\iota \otimes \Psi)(X) = \Psi((\omega \otimes \iota)(X))$$

for all $X \in V \overline{\otimes}_{\mathcal{F}} W_1$ and $\omega \in V_*$. It is easy to see that we have $\|\iota \otimes \Psi\|_{cb} = \|\Psi\|_{cb}$, and if Ψ is a completely isometric isomorphism (respectively, completely contractive projection) then so is $\iota \otimes \Psi$. We call $\iota \otimes \Psi$ an amplification of Ψ , which has been studied by Hamana [15] and Neufang [30].

Now since \mathcal{H}^μ and $\mathcal{H}_\mu = (\mathcal{H}^\mu, \circ)$ have the same predual, the identity map ι_μ is a weak* homeomorphic and completely isometric isomorphism from the weak* closed operator system \mathcal{H}^μ onto the von Neumann algebra \mathcal{H}_μ . So we obtain the weak* homeomorphic and completely isometric isomorphism

$$\iota \otimes \iota_\mu : L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}^\mu \rightarrow L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu.$$

Since both $L_\infty(\mathbb{G})$ and \mathcal{H}_μ are von Neumann algebras, we can identify $L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu$ with the von Neumann algebra $L_\infty(\mathbb{G}) \overline{\otimes} \mathcal{H}_\mu$. We note that since \mathcal{E}_μ is a (not necessarily normal) projection of norm one from $L_\infty(\mathbb{G})$ onto $\mathcal{H}^\mu \subseteq L_\infty(\mathbb{G})$, the map $\iota \otimes \mathcal{E}_\mu$ defines a projection of norm one from $L_\infty(\mathbb{G}) \overline{\otimes} L_\infty(\mathbb{G}) = L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} L_\infty(\mathbb{G})$ onto $L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}^\mu \subseteq L_\infty(\mathbb{G}) \overline{\otimes} L_\infty(\mathbb{G})$ and thus induces a Choi-Effros product such that $L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}^\mu$ becomes a von Neumann algebra. It turns out that (up to the above identification) this von Neumann algebra is exactly equal to $L_\infty(\mathbb{G}) \overline{\otimes} \mathcal{H}_\mu$.

Theorem 7.2. *For any $\mu \in \mathcal{P}_u(\mathbb{G})$ the restriction of Γ to \mathcal{H}_μ induces a left coaction Γ_μ of \mathbb{G} on the von Neumann algebra \mathcal{H}_μ .*

Proof. Let us first show that $\Gamma(\mathcal{H}^\mu) \subseteq L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}^\mu$. Given $x \in \mathcal{H}^\mu$ and $f \in L_1(\mathbb{G})$, we have

$$\Phi_\mu((f \otimes \iota)\Gamma(x)) = (f \otimes \iota)(\iota \otimes \Phi_\mu)\Gamma(x) = (f \otimes \iota)\Gamma(\Phi_\mu(x)) = (f \otimes \iota)\Gamma(x).$$

This shows that $(f \otimes \iota)\Gamma(x)$ is contained in \mathcal{H}^μ for all $f \in L_1(\mathbb{G})$. On the other hand, we clearly have $(\iota \otimes g)\Gamma(x) \in L_\infty(\mathbb{G})$ for all $g \in (\mathcal{H}_\mu)_*$. Hence, $\Gamma(x) \in L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}^\mu$.

It is clear that $\Gamma|_{\mathcal{H}^\mu} : \mathcal{H}^\mu \rightarrow L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}^\mu$ is a normal injective unital completely positive isometry. This induces the map $\Gamma_\mu : \mathcal{H}_\mu \rightarrow L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu$ given by $\Gamma_\mu = (\iota \otimes \iota_\mu) \circ \Gamma \circ \iota_\mu^{-1}$. It suffices to show that Γ_μ is an algebra homomorphism with respect to the corresponding Choi-Effros products on \mathcal{H}_μ and $L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu$. Given $x, y \in \mathcal{H}_\mu$, we now use (3.1) and (3.2) to obtain that

$$\begin{aligned} \Gamma_\mu(x \circ y) &= \Gamma(\mathcal{E}_\mu(xy)) = \Gamma\left(\lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \Phi_\mu^k(xy)\right) = \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n \Gamma(\Phi_\mu^k(xy)) \\ &= \lim_{\mathcal{U}} \frac{1}{n} \sum_{k=1}^n (\iota \otimes \Phi_\mu^k)(\Gamma(xy)) = (\iota \otimes \mathcal{E}_\mu)(\Gamma(x)\Gamma(y)) \\ &= \Gamma_\mu(x) \circ \Gamma_\mu(y) \in L_\infty(\mathbb{G}) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu. \end{aligned}$$

Since Γ is a comultiplication on $L_\infty(\mathbb{G})$, it is clear that Γ_μ satisfies

$$(\iota \otimes \Gamma_\mu) \circ \Gamma_\mu = (\Gamma \otimes \iota) \circ \Gamma_\mu.$$

So Γ_μ defines a left coaction. □

As we discussed above, we have a projection $\mathcal{E}_{\Theta(\mu)}$ of norm one from $\mathcal{B}(L_2(\mathbb{G}))$ onto $\mathcal{H}^{\Theta(\mu)}$ and obtain the Choi-Effros von Neumann algebra product on $\mathcal{H}_{\Theta(\mu)}$ given by

$$X \circ Y = \mathcal{E}_{\Theta(\mu)}(XY)$$

for $X, Y \in \mathcal{H}_{\Theta(\mu)} = (\mathcal{H}^{\Theta(\mu)}, \circ)$. It is easy to see that the restriction of $\mathcal{E}_{\Theta(\mu)}$ to $L_\infty(\mathbb{G})$ is equal to \mathcal{E}_μ and \mathcal{H}_μ is a von Neumann subalgebra of $\mathcal{H}_{\Theta(\mu)}$. Let $\tilde{\Gamma}$ be the comultiplication on $\mathcal{B}(L_2(\mathbb{G}))$ defined in (2.2).

Theorem 7.3. *For any $\mu \in \mathcal{P}_u(\mathbb{G})$ the restriction of $\tilde{\Gamma}$ to $\mathcal{H}_{\Theta(\mu)}$ induces a normal injective unital $*$ -homomorphism*

$$\tilde{\Gamma}_{\Theta(\mu)} : \mathcal{H}_{\Theta(\mu)} \rightarrow \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu$$

between von Neumann algebras. Moreover, the restriction of $\tilde{\Gamma}_{\Theta(\mu)}$ to \mathcal{H}_μ is equal to Γ_μ .

Proof. For any $x \in \mathcal{H}_{\Theta(\mu)}$, we have $\tilde{\Gamma}(x) \in \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes}_{\mathcal{F}} L_\infty(\mathbb{G})$. We can apply (7.1) to get

$$\Phi_\mu((\omega \otimes \iota)\tilde{\Gamma}(x)) = (\omega \otimes \iota)(\iota \otimes \Phi_\mu)\tilde{\Gamma}(x) = (\omega \otimes \iota)(\tilde{\Gamma}(\Theta(\mu)(x))) = (\omega \otimes \iota)(\tilde{\Gamma}(x))$$

for all $\omega \in \mathcal{B}(L_2(\mathbb{G}))_*$. This shows that $\tilde{\Gamma}(\mathcal{H}_{\Theta(\mu)}) \subseteq \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu = \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes}_{\mathcal{F}} \mathcal{H}_\mu$. The rest of proof is similar to that given in the proof of Theorem 7.2. □

For $\mu \in \mathcal{P}_u(\mathbb{G})$, let Γ_μ be the left coaction of \mathbb{G} on the von Neumann algebra \mathcal{H}_μ given in Theorem 7.2. The crossed product $\mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu$ is defined to be the von Neumann algebra

$$\mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu = \{\Gamma_\mu(\mathcal{H}_\mu) \cup (L_\infty(\hat{\mathbb{G}}) \otimes 1)\}''$$

in $\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu$. The following result, which is crucial for us, even holds in the setting of measured quantum groupoids (cf. [9, Theorem 11.6]).

Theorem 7.4. *Let $\mu \in \mathcal{P}_u(\mathbb{G})$. Denote by χ the flip map $\chi(a \otimes b) = b \otimes a$. Then*

$$\beta : x \in \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu \rightarrow (\sigma V^* \sigma \otimes 1)((\chi \otimes \iota)(\iota \otimes \Gamma_\mu)(x))(\sigma V \sigma \otimes 1) \in L_\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu$$

defines a left coaction of \mathbb{G} on the von Neumann algebra $\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu$, and we have

$$\mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu = (\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu)^\beta,$$

where $(\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu)^\beta = \{y \in \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu : \beta(y) = 1 \otimes y\}$ is the fixed point algebra of β .

Now we can prove the main theorem of this section. This was first proved by Izumi [20] for countable discrete groups. He asked in [20, Problem 4.3] whether this could be true for second countable locally compact groups. Izumi's question was first answered affirmatively for second countable groups by Jaworski and Neufang in [22]. Applying to the case $\mathbb{G} = L_\infty(G)$, our theorem shows that this is even true without assuming the second countability. Applied to the case $\mathbb{G} = VN(G)$, our result generalizes the main theorem of [32] where it was assumed either that G has the approximation property, or $\mu \in L_1(\mathbb{G})$.

Theorem 7.5. *Let \mathbb{G} be a locally compact quantum group and let $\mu \in \mathcal{P}_u(\mathbb{G})$. The induced map*

$$\tilde{\Gamma}_{\Theta(\mu)} : \mathcal{H}_{\Theta(\mu)} \rightarrow \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu$$

defines a von Neumann algebra isomorphism between $\mathcal{H}_{\Theta(\mu)}$ and $\mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu$.

Proof. Since $V \in L_\infty(\hat{\mathbb{G}}') \overline{\otimes} L_\infty(\mathbb{G})$, it is easy to see from (2.2) and (7.1) that we have

$$\Theta(\mu)(\hat{x}) \otimes 1 = V^* (\tilde{\Gamma}(\Theta(\mu)(\hat{x}))) V = V^* (\iota \otimes \Phi_\mu)(V(\hat{x} \otimes 1) V^*) V = \hat{x} \otimes 1$$

for all $\hat{x} \in L_\infty(\hat{\mathbb{G}}) = L_\infty(\hat{\mathbb{G}})'$. This shows that $\hat{x} \in \mathcal{H}_{\Theta(\mu)}$ and

$$\hat{x} \otimes 1 = \tilde{\Gamma}(\hat{x}) = \tilde{\Gamma}_{\Theta(\mu)}(\hat{x}) \in \tilde{\Gamma}_{\Theta(\mu)}(\mathcal{H}_{\Theta(\mu)})$$

for all $\hat{x} \in L_\infty(\hat{\mathbb{G}})$. Moreover, since $\mathcal{H}_\mu \subseteq \mathcal{H}_{\Theta(\mu)}$ and the restriction of $\tilde{\Gamma}_{\Theta(\mu)}$ to \mathcal{H}_μ is equal to Γ_μ , we obtain

$$\Gamma_\mu(\mathcal{H}_\mu) = \tilde{\Gamma}_{\Theta(\mu)}(\mathcal{H}_\mu) \subseteq \tilde{\Gamma}_{\Theta(\mu)}(\mathcal{H}_{\Theta(\mu)}).$$

Since $\tilde{\Gamma}_{\Theta(\mu)}$ is a unital $*$ -homomorphism from $\mathcal{H}_{\Theta(\mu)}$ into $\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu$, this implies that

$$\mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu = (\Gamma_\mu(\mathcal{H}_\mu) \cup (L_\infty(\hat{\mathbb{G}}) \otimes 1))'' = \tilde{\Gamma}_{\Theta(\mu)}((\mathcal{H}_\mu \cup L_\infty(\hat{\mathbb{G}})))'' \subseteq \tilde{\Gamma}_{\Theta(\mu)}(\mathcal{H}_{\Theta(\mu)}).$$

Conversely, let $x \in \mathcal{H}_{\Theta(\mu)}$. Then $\tilde{\Gamma}_{\Theta(\mu)}(x) \in \mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu$ and

$$\begin{aligned} \beta(\tilde{\Gamma}_{\Theta(\mu)}(x)) &= (\sigma V^* \sigma \otimes 1)((\chi \otimes \iota)(\iota \otimes \tilde{\Gamma})(\tilde{\Gamma}_{\Theta(\mu)}(x)))(\sigma V \sigma \otimes 1) \\ &= (\sigma V^* \sigma \otimes 1)((\sigma \otimes \iota)(\tilde{\Gamma} \otimes \iota)(\tilde{\Gamma}_{\Theta(\mu)}(x))(\sigma \otimes \iota))(\sigma V \sigma \otimes 1) \\ &= (\sigma V^* \otimes 1)((V \otimes 1)(\tilde{\Gamma}_{\Theta(\mu)}(x)_{13})(V^* \otimes 1))(V \sigma \otimes 1) \\ &= (\sigma \otimes 1)(\tilde{\Gamma}_{\Theta(\mu)}(x)_{13})(\sigma \otimes 1) = 1 \otimes \tilde{\Gamma}_{\Theta(\mu)}(x). \end{aligned}$$

Therefore, we have

$$\tilde{\Gamma}_{\Theta(\mu)}(x) \in (\mathcal{B}(L_2(\mathbb{G})) \overline{\otimes} \mathcal{H}_\mu)^\beta = \mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu$$

for all $x \in \mathcal{H}_{\Theta(\mu)}$. This completes the proof. \square

The following result is an immediate consequence of Theorem 7.5.

Corollary 7.6. *The crossed product von Neumann algebra $\mathbb{G} \ltimes_{\Gamma_\mu} \mathcal{H}_\mu$ is injective.*

By Zimmer's classical result [41], if G is a second countable locally compact group and $\alpha : G \curvearrowright X$ is a measure class preserving action of G on a standard probability space (X, ν) , then the action $\alpha : G \curvearrowright X$ is amenable if and only if the crossed product $\mathbb{G} \ltimes_\alpha L_\infty(X, \nu)$ is injective. This has been generalized to the case of actions of locally compact groups on von Neumann algebras by Anantharaman-Delaroche [1]. Our Corollary 7.6 suggests that a quantum group version of amenable actions can be formulated in the von Neumann algebra setting, and the natural action of the quantum group on its Poisson boundaries is always amenable. We will pursue this idea in detail in a subsequent paper. We finally note that amenable actions of certain discrete quantum groups in the C^* -algebra framework have been considered by Vaes and Vergnoux [39].

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